EXPLANATION

OF THE

"THEORY OF THE CALCULUS."

BY

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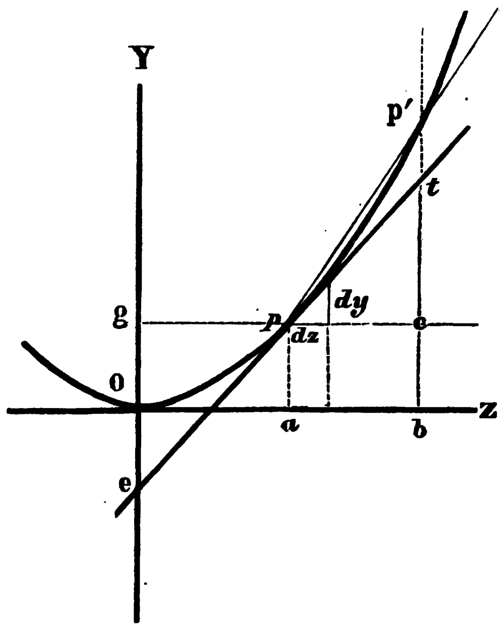
THE writer of this "Explanation" published, several months ago, an octavo volume of ninety pages, entitled "The Theory of the Calculus." He has reason to apprehend, that, partly on account of the new notation invented for it, partly from the novelty of the methods explained in it, and partly, perhaps, from defects in the exposition, his work is found difficult to understand. He takes occasion, therefore, to re-state the substance of what he said in his book; making use, this time, of the ordinary notation, adducing none but familiar notions, and condensing the whole exposition into some eight or ten pages; thus giving, to persons already versed in the Calculus, the means to judge fairly of the merits or demerits of the proposed "Theory." To persons acquainted with the Calculus, this Explanation may very well supersede the published volume. The author presumes that the example he sets, in superseding his own book, is a good one: at all events, he claims it as original.

Let the heavy curve in the figure represent the parabola



whose equation is

$$z^2 = py, \text{ or } y = \frac{z^2}{p}$$

(z being any function whatever of the independent variable x); that is to say, let it represent the parabola whose axis is coincident with the axis of the ys, whose vertex is at the origin of co-ordinates, and whose parameter is p. Through any point, **p**, of the curve, draw a secant, **pp′**, and the tangent **pt**; draw also, through the same point, **pc** parallel to the axis of the zs. The co-ordinates of the point **p** will be z and y (or **gp** and **ap**); and those of the point **p′** will be $z +$ increment of z, and $y +$ increment of y (or **gp** $+$ **pc** and **ap** $+$ **cp**).

It will be perceived, by a reference to the figure, that the curve and its tangent are sensibly coincident for a short distance on each side of the point **p**. Regarding, therefore, the curve as being no real curve, but, on the contrary, a polygon made up of an infinite number of infinitely small rectilinear sides or elements,* and regarding the tangent **pt** as a prolongation of one of these rectilinear elements; making, also, inc. x infinitely small, and denoting it thus, dx (differential of x), which would require us to make inc. y also infinitely small, or dy (differential of y), — we obtain a special infinitely small element of the curve (or one part of that element) located between **p** and the extremity of dy. It will also be observed, since the infinitely small element of the curve and an infinitely small portion of the tangent are (by the hypothesis) absolutely one and the same thing, that the portion of the tangent is just as much coincident with the element of the curve as the element of the curve is coincident with the portion of the tangent. Now, the initiatory problem of the Calculus being to find the value of dy in terms of dz, we may proceed to its solution in one or the other of two obvious ways, since we may regard the distance from **p** to the extremity of dy, either, first, as a portion of the tangent; or, secondly, as an element (or part of an element) of the curve. Regarding this distance as a portion of the tangent, we proceed as follows: We prolong **tp** to **e**, and **cp** to **g**, obtaining the subtangent **ge** equal to twice **gO**, or twice **ap** or $2y$, or (since $y = \frac{z^2}{p}$) equal to $\frac{2z^2}{p}$; for it is proved geometrically, in the books on the Conic Sections, that, *in the case of the parabola,*

* **We shall see, in a moment, how far, if at all, this hypothesis is really admissible.**

the subtangent to the axis is bisected by the vertex. We have, moreover, as is obvious from inspection, by similar triangles,

$$dy : dz :: \mathbf{g\,e} : \mathbf{g\,p};$$

or, because of the property of the parabola,

$$dy : dz :: \frac{2z^2}{p} : z, \text{ or } dy = d\left(\frac{z^2}{p}\right) = \frac{2}{p}zdz.$$

If $p =$ unity, the equation of the curve will be $y = z^2$, and we shall have

$$dy = d(z)^2 = 2zdz.*$$

Before proceeding to the solution of this same problem, and to the finding of this same formula, in the second way,— that is, by regarding the distance between $\mathbf{p}$ and the extremity of dy as an element (or part of an element) of the curve, rather than as a portion of the tangent,—we take the liberty to delay the reader a moment upon simple geometrical and algebraic matters. Afterwards, we will examine the processes that depend upon the fact of curvature, and on the peculiar methods of the Infinitesimal Calculus.

We suppose the reader will readily admit as obvious, that $d(u + v)$ is equal to $du + dv$, not only under the ordinary hypothesis, but also in the case that the distance from the point of tangency to the extremity of $d(u + v)$ is to be regarded, in analogy with the course just followed, as a part of the tangent to the curve $w = u + v$, rather than as an element (or part of an element) of the curve itself. If he will not, he will find the fact fully proved in our published book.

These two formulas,

$$d\left(\frac{z^2}{p}\right) = \frac{2}{p}zdz, \text{ and } d(u + v) = du + dv,$$

* This demonstration is geometrical, consequently irrefragable, and therefore to be preferred to any transcendental demonstration based on infinitesimal methods.

enable us to obtain, by simple algebraic methods, the differentials of all algebraic expressions.

EXAMPLE 1.—Let it be required to find the differential of the difference $u - v$.

$$\text{Put} \quad u - v = w, \quad \text{then} \quad u = w + v;$$
$$\text{and} \quad du = dw + dv, \text{ or } dw = du - dv;$$
$$\text{or} \quad d(u - v) = du - dv.$$

EXAMPLE 2.—Let it be required to find the differential of the sum $u - v - w$.

$$\text{Put} \quad u - v = q;$$
$$\begin{aligned} \text{then} \quad d(q - w) &= dq - dw, \\ &= d(u - v) - dw, \\ &= du - dv - dw, \end{aligned}$$
$$\text{or} \quad d(u - v - w) = du - dv - dw.$$

EXAMPLE 3.—Let it be required to find the differential of the product $u \times v$.

Put $u + v = w$, then $d(u + v) = du + dv = dw$.

Also, since

$$u^2 + 2uv + v^2 = w^2, \text{ or } 2uv = w^2 - u^2 - v^2,$$
$$uv = \frac{w^2}{2} - \frac{u^2}{2} - \frac{v^2}{2}.$$

We have, therefore, by Ex. 2, and by the formula

$$d\left(\frac{z^2}{p}\right) = \frac{2}{p} z dz,$$
$$\begin{aligned} d(uv) &= wdw - udu - vdv, \\ &= (u + v)\, d(u + v) - udu - vdv, \\ &= udu + udv + vdu + vdv - udu - vdv, \\ &= udv + vdu. \end{aligned}$$

The formula sought is, therefore, $d(uv) = udv + vdu$.

EXAMPLE 4.—Let it be required to find the differential of the product $u \times v \times w$.

$$\begin{aligned} \text{Put} \qquad u \times v &= q; \\ \text{then} \qquad d(qw) &= qdw + wdq, \\ &= uvdw + wd(u \times v), \\ &= uvdw + w(udv + vdu), \\ &= uvdw + wudv + wvdu. \end{aligned}$$

EXAMPLE 5.—Let it be required to find the differential of u^3, or of $u \times u \times u$.

$$\begin{aligned} \text{We have} \quad d(u \times u \times u) &= u^2du + u^2du + u^2du, \\ \text{or} \qquad d(u^3) &= 3u^{3-1}du. \end{aligned}$$

It is useless to multiply these examples. The reader can furnish the others from his own resources: at all events, he may find all the special cases noticed, and worked out, in the published book.

The second, and less expeditious, way of finding the formula

$$d\left(\frac{z^2}{p}\right) = \frac{2}{p}zdz,$$

the one depending upon a consideration of the fact of curvature, and on the use of the infinitesimal method, is, although less satisfactory, not much more difficult. The ordinate **b p′** is (see the figure) $y + \text{inc.}\, y$: and we have, by the equation of the curve,

$$y + \text{inc.}\, y = \frac{1}{p}(z + \text{inc.}\, z)^2 = \frac{1}{p}[z^2 + 2z \text{ inc.}\, z + (\text{inc.}\, z)^2];$$

or, since $y = \frac{1}{p}(z^2)$,

$$\text{inc.}\, y = \frac{1}{p}[2z \text{ inc.}\, z + (\text{inc.}\, z)^2];$$

which last equation may be put under the form

$$\frac{\text{inc.}\, y}{\text{inc.}\, z} = \frac{1}{p}(2z + \text{inc.}\, z).$$

If, now, we imagine inc. z to continuously diminish until it becomes infinitely small, we shall imagine the point **p′** to run continuously along the curve in the direction from **p′** to **p**; and the secant **pp′** to revolve continuously, in the plane of the paper, on the point **p** as a pivot, towards coincidence with the tangent **pt**. When inc. z becomes infinitely small (or dz), inc. y will also be infinitely small (or dy); the secant **pp′** will not be sensibly distinguishable from the tangent **pt**; and our last equation will present itself in the form

$$\frac{dy}{dz} = \frac{1}{p}(2z + dz), \text{ or } dy = \frac{1}{p}[2zdz + (dz)^2].$$

But, since dz is infinitely small, neither it nor its square is really competent to add any thing to the second member of the equation: the last term may, therefore, be stricken out as comparatively valueless; giving us

$$\frac{dy}{dz} = \frac{2z}{p}, \text{ or } dy = \frac{2zdz}{p},$$

the formula already obtained.

It must be observed, however, that the secant is not really revolved into coincidence with the tangent until inc. $z = 0$, and inc. $y = 0$, causing the equation $\frac{\text{inc.}\, y}{\text{inc.}\, z} = \frac{1}{p}(2z + \text{inc.}\, z)$ to present itself in the singular form $\frac{0}{0} = \frac{1}{p}(2z + 0)$; and it must be further noticed, that these zeros are absolute, and that the expression $\frac{\text{absolute } 0}{\text{absolute } 0}$ is hard to interpret. When we examine the figure, we are led to suspect that $dy = \frac{1}{p}(2zdz)$ fails to reach the curve, and by precisely the quantity $\frac{(dz)^2}{p}$ that has been neglected. It is, in fact, proved in the preceding

articles, geometrically, that $dy = \frac{1}{p}(2zdz)$ reaches to the tangent, and no farther; and it is proved here that

$$\frac{1}{p}[2zdz + (dz)^2]$$

reaches beyond the tangent to the curve. It follows, therefore, that the theory is demonstrably false which teaches that a curve is a polygon made up of an infinite number of infinitely small sides, or elements; it follows, also, that the statement that a tangent to a curve is the prolongation of one of the curve's infinitely small elements, is a statement void of meaning. This conclusion may be confirmed, if confirmation be called for, by the reflection that the radius drawn from the centre of a circle to the extremity of any one of its chords, is necessarily longer than the perpendicular let fall from the centre on the middle of that chord; showing that, if the chord have any actual length whatever, infinitely small or other, the middle of the chord cannot, by any possibility, be a point of the circumference.

It is evident that the equation $\frac{\text{inc.}\, y}{\text{inc.}\, z} = \frac{1}{p}(2z + \text{inc.}\, z)$ cannot assume the singular form $\frac{0}{0} = \frac{1}{p}(2z + 0)$ until the two points **p** and **p′** actually coincide, and until the secant becomes absolutely one and the same line with the tangent. The expression $\frac{0}{0}$ seems here to denote, not absurdity, nor even indetermination, but simply the self-stultification of the reasoner who draws conclusions from a fact of secancy which he has himself imposed, and then affirms that his conclusions hold good after he has, by his own act, caused the fact of secancy, on which the validity of his reasoning depends, to no longer exist.

The only fundamental innovation that we make upon the ordinary point of view is really this: —

We affirm that dy is the infinitely small increment of the ordinate of *the special tangent* that passes through the point p, and not at all the increment of the ordinate of the curve. We affirm, consequently, that the reason why $d(x^2)$ is always $2xdx$, and never $2xdx + (dx)^2$, is simply because $d(x^2)$ never reaches far enough to meet the curve $y = x^2$. We take occasion to remark, that the expression, "two consecutive points of a curve," is utterly void of meaning so long as we adhere to the definition of the word "point" that is given in the elementary books.

In the Infinitesimal Calculus, $d(x)^2$ is really $2xdx$, and not $2xdx + (dx)^2$, for the very reason that we assign, although the text-books say the contrary.

The increment, inc. y, of the ordinate of the curve $y = x^2$, is $2x$ inc. x + (inc. x)^2; and the increment of the ordinate of the tangent is $2x$ inc. x. Neglecting the term (inc. x)^2, we pass from the consideration of the curve to the consideration of the tangent; that is to say, we pass from the consideration of varying variation to the consideration of uniform variation. We write, not to condemn the infinitesimal method, but to correct it, and to confirm it. It is the true method in investigations of physical science; but it is a method (at least in the form under which it is set forth in the books) not fit to be used in teaching intelligent boys who are just beginning to face the difficulties of the Calculus.

The reader will naturally stop us here, and object to our simple geometrical processes, saying that our method may be all well enough in the management of algebraic functions, but that it must inevitably break down in the presence of tran-

scendental expressions. We will show him, by one or two easy examples, that the contrary is the case.

EXAMPLE 1. — It has been proved by Huygens and others, geometrically, that, in the logarithmic curve, the subtangent (to the axis of the logarithms) is constant. Drawing the curve $y = \log. z$, or $a^y = z$ (z being the number, and any function whatever of the independent variable x), and drawing a tangent to this curve through one of its points, p, we have, by similar triangles (denoting the constant subtangent by m),

$$dy : dz :: m : z, \text{ or } dy = \frac{mdz}{z},$$

$$\text{or } d(\log. z) = \frac{mdz}{z}.$$

This formula, which will be at once recognized as the right one, becomes available as soon as the nature and value of m are determined; and we have shown in our book that the determination of the nature and value of m is perfectly feasible by processes independent of the infinitesimal method.

EXAMPLE 2. — Draw the circle whose equation is

$$y^2 = 2Rz - z^2.$$

Through the extremity of any ordinate, y, draw a radius, and a tangent perpendicular to the radius.

Now, we have proved in our book, by strictly geometrical processes, that, if the arc of the circle $y^2 = 2Rz - z^2$ be denoted by u, we shall have

$$du = \sqrt{(dy)^2 + (dz)^2}.$$

We have, then, as is obvious from an inspection of the figure, by similar triangles,

$$dy : du :: \cos. u : R, \text{ or } dy = \frac{R}{\cos. u\, du}.$$

But y is $\sin. u$: therefore

$$d(\sin. u) = \frac{\cos. u\, du}{R}.$$

The application of the method to the various cases of the circular functions is now easy.

We might adduce, by way of a third example, the use of our method in the finding of differentials when curves are given by their polar co-ordinates; but, although we have worked out the various cases, we do not state them here (and we have not mentioned them in our book), partly on account of lack of space, and partly because we prefer to leave the investigation to the reader's ingenuity.

We have explained, in our book, the theory, from our own point of view, of tangency and curvature, of problems *de maximis et minimis*, of multiple points and cusps, of points of contrary flexure, of the radius of curvature, of involutes and evolutes, of the rectification of curved lines, of the planification of curved surfaces, of the measure of solids of revolution, of the formulas of McLaurin and Taylor, &c., &c., — all in a volume of less than a hundred pages (large print, and leaded); showing that, at the least, our method does not involve prolixity of exposition. But, after what has been said above, the intelligent reader will require no aid from us in regard to any of these matters.

In pages 17, 18, 19, and 20, of the book, the author wastes his time in demolishing a gratuitous difficulty, or rather mare's-nest, of his own private creation: he allowed himself to be bewildered and deceived by his own notation; and did not discover, until he saw the book in print, the utter aimlessness of what he was trying to say. The reader is respectfully requested, if he consult the book at all, to skip over

those four pages. What the author was trying to say in that place, he has actually said in the beginning of this "Explanation."

We will remark, in conclusion, that, in our book, we define a straight line tangent to a curve, as a straight line touching the curve, and located, on both sides of the point of contact, opposite the convex side of the curve. This definition is axiomatic, and (it is presumed) justifies itself.

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